

UNIQUENESS OF REDUCED ALTERNATING RATIONAL 3-TANGLE DIAGRAMS

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ABSTRACT. Tangles were introduced by J. Conway. In 1970, he proved that every rational 2-tangle defines a rational number and two rational 2-tangles are isotopic if and only if they have the same rational number. So, from Conway's result we have a perfect classification for rational 2-tangles. However, there is no similar theorem to classify rational 3-tangles.

In this paper, we introduce an invariant of rational n -tangles which is obtained from the Kauffman bracket. It forms a vector with Laurent polynomial entries. We prove that the invariant classifies the rational 2-tangles and the reduced alternating rational 3-tangles. We conjecture that it classifies the rational 3-tangles as well.

1. INTRODUCTION

A n -tangle is the disjoint union of n properly embedded arcs in the unit 3-ball. A *rational n -tangle* is a n -tangle $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ in a 3-ball B^3 such that there exists a homeomorphism of pairs $\Phi : (B^3, \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) \longrightarrow (D^2 \times I, \{p_1, p_2, \dots, p_n\} \times I)$. Then two rational n -tangles, T, T' , in B^3 are *isotopic*, denoted by $T \approx T'$, if there is an orientation-preserving self-homeomorphism $h : (B^3, T) \rightarrow (B^3, T')$ that is the identity map on the boundary.

H. Cabrera-Ibarra [3] found a pair of invariants which is defined for all rational 3-tangles. Each invariant is a 3×3 matrix with complex number entries. The pair of invariants classifies the elements of six special sets of rational 3-tangles each of which contains the braid 3-tangles. However, it does not cover the collection of all alternating rational 3-tangles.

Also, Emert and Ernst [5] classified the collection of *essential* alternating rational n -tangles which is a set of alternating rational n -tangles satisfying a certain condition. However, it also does not cover the collection of all alternating rational 3-tangles.

In this paper, I would like to classify the collection of all alternating rational 3-tangles. With a similar argument, it could be possible to classify the collection of all alternating rational n -tangles.

Let S^2 be a sphere smoothly embedded in S^3 and let K be a link transverse to S^2 . The complement in S^3 of S^2 consists of two open balls, B_1 and B_2 . We assume that S^2 is xz -plane $\cup \{\infty\}$. Now, consider the projection of K onto the flat xy -plane. Then, the projection onto the xy -plane of S^2 is the x -axis and B_1 projects to the upper half plane and B_2 projects to the lower half plane. The projection gives us a *link diagram*, where we make note of over and undercrossings. The diagram of the link K is called a *plat on $2n$ -strings*, denoted by $p_{2n}(w)$, if it is the union of a $2n$ -braid w and $2n$ unlinked and unknotted arcs which connect pairs

of consecutive strings of the braid at the top and at the bottom endpoints and S^2 meets the top of the $2n$ -braid. (See the first and second diagrams of Figure 1.) Any link K in S^3 admits a plat presentation. i.e., K is ambient isotopic to a plat ([2], Theorem 5.1). The bridge (plat) number $b(K)$ of K is the smallest possible number n such that there exists a plat presentation of K on $2n$ strings. We say that K is n -bridge link if the bridge number of K is n . We remark that the braid group \mathbb{B}_{2n} is generated by $\sigma_1, \sigma_2, \dots, \sigma_{2n-1}$ which are twisting of two adjacent strings. For example, $w = \sigma_2^{-1} \sigma_4^{-1} \sigma_3 \sigma_1^3 \sigma_5^2 \sigma_4^{-1} \sigma_2^{-1}$ is the word for the 6 braid of the first diagram of Figure 1.

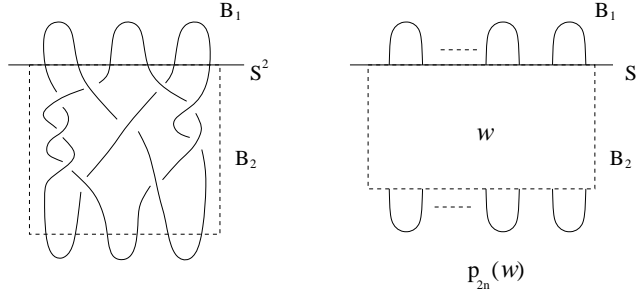


FIGURE 1.

Then we say that a plat presentation is *standard* if the $2n$ -braid w of $p_{2n}(w)$ involves only $\sigma_2, \sigma_3, \dots, \sigma_{2n-1}$.

We remark that $K \cap B_2$ is a rational tangle.

Now, we define a *2n-plat presentaion* for rational n -tangles $K \cap B_2$ in B_2 , denoted by $q_{2n}(w)$, if it is the union of a $2n$ -braid w and n unlinked and unknotted arcs which connect pairs of strings of the braid at the bottom endpoints with the same pattern as in a plat presentation for a link and the projection of ∂B_2 onto the flat xy -plane meets the top of the $2n$ -braid.

We note that $q_{2n}(w)$ is a rational n -tangle in B_2 since we can obtain a trivial rational n -tangle from the rational n -tangle by a sequence of half Dehn twists which are automorphisms of B^3 that preserve the six punctures.

We also say that $\overline{q_{2n}(w)} (= p_{2n}(w))$ is the *plat closure* of $q_{2n}(w)$ if it is the union of $q_{2n}(w)$ and n unlinked and unknotted arcs in B_1 which connect pairs of consecutive strings of the braid at the top endpoints.

The tangle diagrams with the circles in Figure 2 give the diagrams of trivial rational 2, 3-tangles as in [1], [3], [6], [8]. The right sides of each pair of diagrams show the trivial rational 2, 3-tangles in B_2 .

A tangle diagram TD (or $2n$ -plat presentation) is *reduced* alternating if TD is alternating and TD does not have a self-crossing which can be removed by a Type I Reidemeister move.

We note that $q_4(w)$ is alternating if and only if $\overline{q_4(w)}$ is alternating, possibly not reduced alternating.

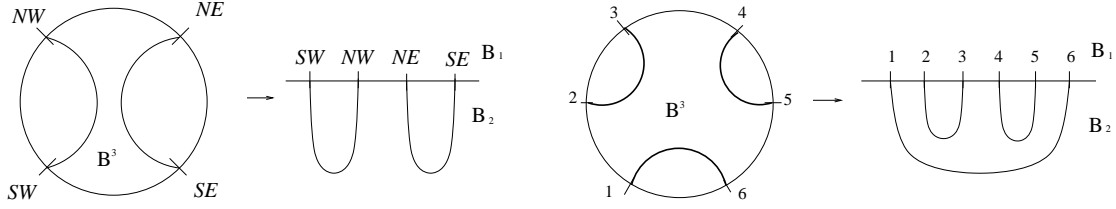


FIGURE 2.

In section 2, we introduce the Kauffman bracket of a rational tangle diagram and discuss how to calculate it. Also, we define a vector from the Kauffman bracket of a rational 2-tangle diagram.

Then, we will prove that the vector from the Kauffman bracket of a rational 2-tangle diagram is an invariant which can classify the rational 2-tangles in section 3.

Finally, we will show that the vector from the Kauffman bracket of a rational 3-tangle diagram is an invariant of rational 3-tangles and especially it classifies the reduced alternating rational 3-tangles in section 4.

2. THE KAUFFMAN BRACKET AND ITS CALCULATION

Let $\Lambda = \mathbb{Z}[a, a^{-1}]$ and L be a link diagram. I want to emphasize here that K, L and T stand for a diagram and \mathbb{K}, \mathbb{L} and \mathbb{T} stand for a knot or link for convenience.

We recall that the Kauffman bracket $\langle L \rangle \in \Lambda$ of L is obtained from the three axioms

$$\begin{aligned} \langle \bigcirc \rangle &= 1 \\ \langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rangle &= a \langle \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \rangle + a^{-1} \langle \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \rangle \\ \langle L \cup \bigcirc \rangle &= k \langle L \rangle, \text{ where } k = -a^2 - a^{-2} \end{aligned}$$

The symbol $\langle \rangle$ indicates that the changes are made to the diagram locally, while the rest of the diagram is fixed.

The Kauffman polynomial $X_L(a) \in \Lambda$ is defined by

$$X_L(a) = (-a^{-3})^{w(\vec{L})} \langle L \rangle,$$

where the writhe $w(\vec{L}) \in \mathbb{Z}$ is obtained by assigning an orientation to L , and taking a sum over all crossings of L of their indices e , which are given by the following rule

$$e(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}) = 1, \quad e(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}) = -1.$$

Then we have the following theorem.

Theorem 2.1. ([8]) *If \mathbb{K} is a 2-bridge link, then there exists a word w in \mathbb{B}_4 so that the plat presentation $p_4(w)$ is reduced alternating and standard and represents a link isotopic to \mathbb{K} .*

Since $p_4(w)$ is standard, we consider \mathbb{B}_3 instead of \mathbb{B}_4 . Then, let σ_1 and σ_2 be the two generators of \mathbb{B}_3 . I want to emphasize here that we are changing from σ_2 and σ_3 to σ_1 and σ_2 .

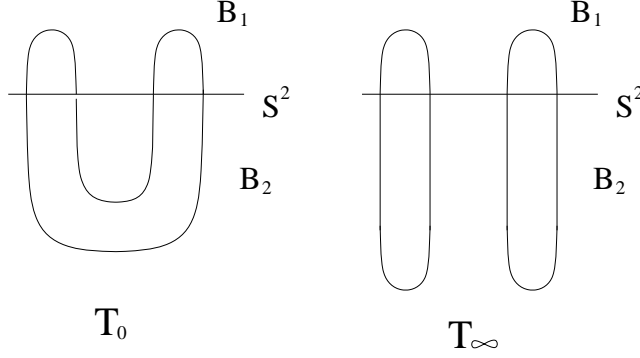


FIGURE 3.

Goldman and Kauffman [5] define the *bracket polynomial* of the two tangle diagram T as $\langle T \rangle = f(a) \langle T_0 \rangle + g(a) \langle T_\infty \rangle$, where the coefficients $f(a)$ and $g(a)$ are Laurent polynomials in a and a^{-1} that are obtained by starting with T and using the three axioms repeatedly until only the two trivial tangle diagrams T_0 and T_∞ in the expression given for T are left. We note that $f(a)$ and $g(a)$ are invariant under regular isotopy of T , where regular isotopy is the equivalence relation of link diagrams that is generated by using the 2nd and 3rd Reidemeister moves only. So, we define the coefficients vector $(f(a), g(a))$ which is a regular invariant of the rational 3-tangles.

Let $\mathcal{A} = \langle T_0 \rangle$ and $\mathcal{B} = \langle T_\infty \rangle$. So, $\langle T \rangle = f(a)\mathcal{A} + g(a)\mathcal{B}$.

We assume that \mathbb{T} is a reduced alternating rational 2-tangle. Then we have $w = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdots \sigma_1^{\epsilon_{2k-1}}$ for some positive (negative) integers ϵ_i for $2 \leq i \leq 2k-1$ and non-negative (non-positive) integer ϵ_1 for \mathbb{T} . We note that w needs to end at $\sigma_1^{\pm 1}$. If $w = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdots \sigma_1^{\epsilon_{2k-1}} \sigma_2^{\epsilon_{2k}}$ for some positive (negative) integer ϵ_{2k} then $q_4(w)$ is not a reduced alternating tangle diagram. i.e., the tangle diagram has a self crossing which can be removed by the first Reidemeister move.

Suppose that $T_1 = q_4(w)$, where $w = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdots \sigma_1^{\epsilon_{2n-1}}$ for some positive (negative) integers ϵ_i ($2 \leq i \leq 2n-1$) and non-negative (non-positive) integer ϵ_1 .

$$\text{Let } A_1^{\pm 1} = \begin{pmatrix} -a^{\mp 3} & a^{\mp 1} \\ 0 & a^{\pm 1} \end{pmatrix} \quad \text{and} \quad A_2^{\pm 1} = \begin{pmatrix} a^{\pm 1} & 0 \\ a^{\mp 1} & -a^{\mp 3} \end{pmatrix}.$$

Also, let $A = A_1^{\epsilon_1} A_2^{-\epsilon_2} \cdots A_1^{\epsilon_{2n-1}}$.

Then we can calculate the two coefficients $f(a)$ and $g(a)$ of \mathcal{A} and \mathcal{B} of T_1 as follows.

Theorem 2.2. (*Eliahou-Kauffman-Thistlethwaite [4]*) Suppose that $T_1 (=q_4(u))$ is a plat presentation of a rational 2-tangle \mathbb{T}_1 which is alternating and standard so that $u = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdots \sigma_1^{\epsilon_{2n-1}}$ for some positive (negative) integers ϵ_i ($1 \leq i \leq 2n-1$) and non-negative (non-positive) integer ϵ_1 . Then,

$\langle T_1 \rangle = f(a)\mathcal{A} + g(a)\mathcal{B}$, where $f(a)$ and $g(a)$ are given by

$$(f(a), g(a))^t = A \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ i.e., the second column of } A, \text{ where } A = A_1^{\epsilon_1} A_2^{-\epsilon_2} \cdots A_1^{\epsilon_{2n-1}}.$$

Then we show the following lemma which helps us to show Theorem 3.2.

Lemma 2.3. Let $A = A_2^m$ for some non-zero integer m .

$$\text{Then } A = \begin{pmatrix} a^m & 0 \\ \frac{a^{m+2} + (-1)^{m+1}a^{2-3m}}{1+a^4} & (-1)^m a^{-3m} \end{pmatrix}.$$

Proof. First, assume that m is a positive integer.

$$\text{Let } A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

I will use induction on m to show this lemma.

We can easily check that if $m = 1$ then $b_{11} = a, b_{12} = 0, b_{21} = a^{-1}$ and $b_{22} = -a^{-3}$ from A_2 .

Suppose that the claim is true when $m = k$. i.e., $b_{11} = a^k, b_{12} = 0, b_{22} = (-1)^k a^{-3k}$, and

$$b_{21} = \frac{a^{k+2} + (-1)^{k+1}a^{2-3k}}{1+a^4}.$$

$$\text{Now, consider } A_2^{k+1} = A_2 A_2^k = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix}.$$

Then we have $b'_{11} = aa^k = a^{k+1}, b'_{12} = 0, b'_{22} = -a^{-3}(-1)^k a^{-3k} = (-1)^{k+1} a^{-3(k+1)}$ and

$$\begin{aligned} b'_{21} &= a^{-1}(a^k) - a^{-3} \left(\frac{a^{k+2} + (-1)^{k+1}a^{2-3k}}{1+a^4} \right) = a^{k-1} - \frac{a^{k-1} + (-1)^{k+1}a^{-1-3k}}{1+a^4} \\ &= \frac{a^{k-1}(1+a^4) - (a^{k-1} + (-1)^{k+1}a^{-1-3k})}{1+a^4} = \frac{a^{k+3} + (-1)^{k+2}a^{-1-3k}}{1+a^4} = \frac{a^{(k+2)+1} + (-1)^{(k+1)+1}a^{2-3(k+1)}}{1+a^4}. \end{aligned}$$

This completes the proof of the case when m is a positive integer.

Similarly, we can show the case when m is a negative integer. □

3. A NEW INVARIANT OF RATIONAL 2-TANGLES

For a Laurent polynomial $p(a)$, let $M(p(a))$ be the maximal power of a in $p(a)$ and $m(p(a))$ be the minimal power of a in $p(a)$. We say that a link K is *connected* if no component of K

is split. Similarly, we say that a tangle T is *connected* if no component of T is split.

K. Murasugi [12] proved the following theorem and it helps to prove Theorem 3.2.

Theorem 3.1 ([12]). *Suppose that K is a connected reduced alternating projection of an alternating link \mathbb{K} . Then $(M(X_K) - m(X_K))/4$ is the number of crossings of K .*

We recall that there is a perfect classification of rational 2-tangles which define a rational number. The following theorem tells us the coefficients vector of rational 2-tangle diagrams classifies the corresponding rational 2-tangles. However, it is not good as the classical invariant.

Theorem 3.2. *Suppose that T is a tangle diagram of a rational 2-tangle \mathbb{T} so that $\langle T \rangle = f(a) \langle T_0 \rangle + g(a) \langle T_\infty \rangle$.*

Then, $\mathbb{T} \approx \mathbb{T}_\infty$ if and only if $(f(a), g(a)) = (-a^{-3})^k(0, 1)$ for some integer k , where \mathbb{T}_∞ is the tangle with the diagram T_∞ .

Proof. First, we suppose that $\mathbb{T} \approx \mathbb{T}_\infty$.

Since $\mathbb{T} \approx \mathbb{T}_\infty$, we get T_∞ from T after applying a sequence of a finite number of the three Reidemeister moves. We note that $f(a)$ and $g(a)$ are invariant under regular isotopy of T .

Now, consider the first Reidemeister moves as in Figure 4.

$$\begin{aligned}
 \langle \text{crossing} \rangle &= a \langle \text{cup} \rangle + a^{-1} \langle \text{cap} \rangle = -a^{-3} \langle \text{cup} \rangle \\
 \langle \text{crossing} \rangle &= a \langle \text{cap} \rangle + a^{-1} \langle \text{cup} \rangle = -a^3 \langle \text{cup} \rangle
 \end{aligned}$$

FIGURE 4.

This implies that $(f(a), g(a)) = (-a^{-3})^k(0, 1)$ for some integer k since $\langle T_\infty \rangle = 0 \cdot \langle T_0 \rangle + 1 \cdot \langle T_\infty \rangle$.

In order to show the opposite direction, assume that there is a non-trivial reduced alternating projection T so that $\langle T \rangle = (-a^{-3})^k \langle T_\infty \rangle$.

Let $q_4(w)$ be the plat presentation for T which is standard and reduced alternating.

Then, either $w = \sigma_1^{\epsilon_1}$ for a non-zero integer ϵ_1 or $w = \sigma_2^{-\epsilon_0} \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \sigma_1^{\epsilon_3} \cdots \sigma_1^{\epsilon_{2n-1}}$ for some positive (negative) integers ϵ_i ($1 \leq i \leq 2n-1$) and non-negative (non-positive) integer ϵ_0 .

If $w = \sigma_1^{\pm 1}$, then we see that $(f(a), g(a)) = (a^{\mp 1}, a^{\pm 1}) \neq (-a^{-3})^k(0, 1)$ for any k .

If $w = \sigma_1^{\epsilon_1}$ for $|\epsilon_1| \geq 2$, then we have $p_4(w)$ which represents a reduced alternating link \mathbb{K} having a diagram $K = \overline{T}$. So, we note that the Kauffman polynomial of K is not trivial. However, we should have trivial Jones polynomial for K since $\langle T \rangle = (-a^{-3})^k(0, 1)$. This contradicts the assumption.

If $w = \sigma_2^{-\epsilon_0} \sigma_1$ for non-negative integer ϵ_0 , then $A = A_2^{-\epsilon_0} A_1$. By using Lemma 3.2, we have

$$(f(a), g(a)) = \left(a^{-\epsilon_0-1}, \frac{a^{-\epsilon_0+1} + (-1)^{-\epsilon_0} a^{3\epsilon_0+5}}{1 + a^4} \right) \neq (-a^{-3})^k(0, 1) \text{ for any } k.$$

Similarly, if $w = \sigma_2^{-\epsilon_0} \sigma_1^{-1}$ for non-positive integer ϵ_0 then we have

$$(f(a), g(a)) = \left(a^{-\epsilon_0+1}, \frac{a^{-\epsilon_0+3} + (-1)^{-\epsilon_0} a^{3\epsilon_0-1}}{1 + a^4} \right) \neq (-a^{-3})^k(0, 1) \text{ for any } k.$$

If $w = \sigma_2^{-\epsilon_0} \sigma_1^{\epsilon_1}$ for $\epsilon_1 \geq 2$, then we have $p_4(w)$ which represents a non-trivial alternating link. So, this violates the assumption too.

If $w = \sigma_2^{-\epsilon_0} \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \sigma_1^{\epsilon_3} \cdots \sigma_1^{\epsilon_{2n-1}}$ for some positive (negative) integers ϵ_i ($1 \leq i \leq 2n-1$, $n \geq 2$) and non-negative (non-positive) integer ϵ_0 , then we have the reduced alternating presentation $v = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdots \sigma_1^{\epsilon_{2n-1}}$ for $p_4(v)$ for some positive (negative) integers ϵ_i ($1 \leq i \leq 2n-1$, $n \geq 2$).

This implies that $M(X_K) - m(X_K) \geq 4$ by Theorem 3.1.

However, if $(f(a), g(a)) = (0, (-a^{-3})^k)$ then $M(X_K) - m(X_K) = 0$ by Theorem 3.1.

This also contradicts the assumption.

Therefore, if $\langle T \rangle = (-a^{-3})^k \langle T_\infty \rangle$ then $\mathbb{T} \approx \mathbb{T}_\infty$.

This completes the proof. □

Corollary 3.3. *Suppose that T and T' are the projections onto the xy -plane of two rational 2-tangles \mathbb{T} and \mathbb{T}' in B^3 so that $\langle T \rangle = f(a) \langle T_0 \rangle + g(a) \langle T_\infty \rangle$ and $\langle T' \rangle = f'(a) \langle T_0 \rangle + g'(a) \langle T_\infty \rangle$ respectively.*

Then, $\mathbb{T} \approx \mathbb{T}'$ if and only if $(f'(a), g'(a)) = (-a^{-3})^k(f(a), g(a))$ for some integer k .

Proof. Let $q_4(w)$ be the plat presentation for T and $q_4(v)$ be the plat presentation for T' . Now, consider $e = w^{-1}w$ and $w^{-1}v$.

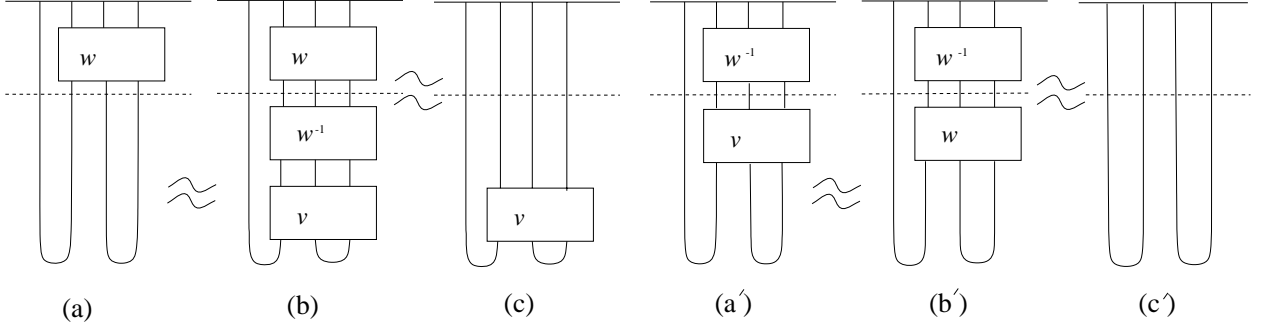


FIGURE 5.

Let \mathbb{T}_1 and \mathbb{T}'_1 be the rational 2-tangles of the plat presentations $q_4(e)$ and $q_4(w^{-1}v)$ respectively. We note that $\mathbb{T}_1 \approx \mathbb{T}_\infty$.

Then by Theorem 3.2, $\mathbb{T}'_1 \approx \mathbb{T}_1 \approx \mathbb{T}_\infty$ if and only if $\langle T'_1 \rangle = (-a^{-3})^k \langle T_\infty \rangle$ for some k .

Now, we claim that $\langle T'_1 \rangle = (-a^{-3})^k \langle T_\infty \rangle$ for some k if and only if $\langle T' \rangle = (-a^{-3})^{k'} \langle T \rangle$ for some k' . To prove this, we repeatedly use the three axioms to remove the crossings below the dotted line of the diagram (a) and the diagram (b) respectively to have $\langle T'_1 \rangle = (-a^{-3})^k \langle T_\infty \rangle$ for some k . Then we repeatedly use the three axioms to remove the crossings above the dotted line of the diagram (a) and the diagram (b) respectively. (Refer to the diagram (a) and (b) of Figure 5.) Since $v = ww^{-1}v$, we note that if $\langle T'_1 \rangle = (-a^{-3})^k \langle T_\infty \rangle$ for some k then $\langle T' \rangle = (-a^{-3})^{k'} \langle T \rangle$ for some k' . Similarly, we note that if $\langle T' \rangle = (-a^{-3})^k \langle T \rangle$ for some k then $\langle T'_1 \rangle = (-a^{-3})^{k'} \langle T_\infty \rangle$ for some k' by using the diagrams (a'), (b') and (c') since $e = w^{-1}w = w^{-1}v$.

Now, it is enough to show that $\mathbb{T}_1 \approx \mathbb{T}'_1$ if and only if $\mathbb{T} \approx \mathbb{T}'$. This is also proved by the diagrams of Figure 5.

Therefore, $\mathbb{T} \approx \mathbb{T}'$ if and only if $\langle T' \rangle = (-a^{-3})^{k'} \langle T \rangle$ for some k' .

This completes the proof. □

4. A NEW INVARIANT OF RATIONAL 3-TANGLES

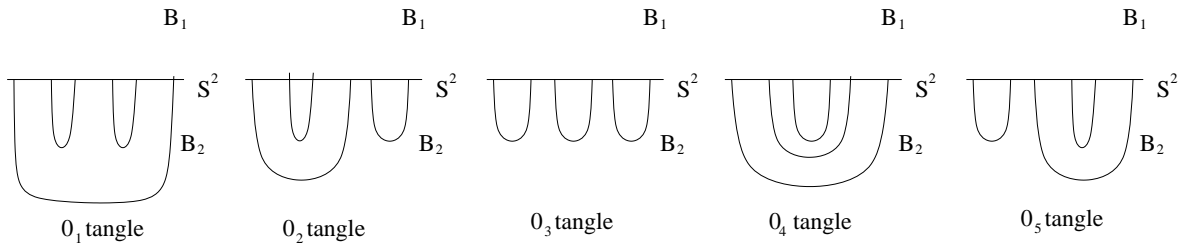


FIGURE 6.

Now, consider a rational 3-tangle \mathbb{T} . Let T be a rational 3-tangle diagram.

H. Cabrera-Ibarra [3] defined the bracket polynomial of the rational 3-tangle diagram T of \mathbb{T} as $\langle T \rangle = f_1^T(a) \langle 0_1 \rangle + f_2^T(a) \langle 0_2 \rangle + f_3^T(a) \langle 0_3 \rangle + f_4^T(a) \langle 0_4 \rangle + f_5^T(a) \langle 0_5 \rangle$, where $f_i^T(a)$ are Laurent polynomials in a and a^{-1} that are obtained by starting with T and using the three axioms repeatedly until only the five trivial tangle diagrams $\langle 0_j \rangle$ in the expression given for T are left. (See Figure 6.)

Then, we define the vector $v_T = (f_1^T(a), f_2^T(a), \dots, f_5^T(a))$ for a rational 3-tangle diagram T . Then we note that the vector v_T is an invariant under regular isotopy of T . Especially we have the following theorem.

Theorem 4.1. *If $\mathbb{T} \approx \mathbb{T}'$ then $v_T = (-a^{-3})^k v_{T'}$ for some k , where T is a tangle diagram of \mathbb{T} and T' is a tangle diagram of \mathbb{T}' .*

Proof. It is the generalization of the proof for the one direction of Theorem 3.2. \square

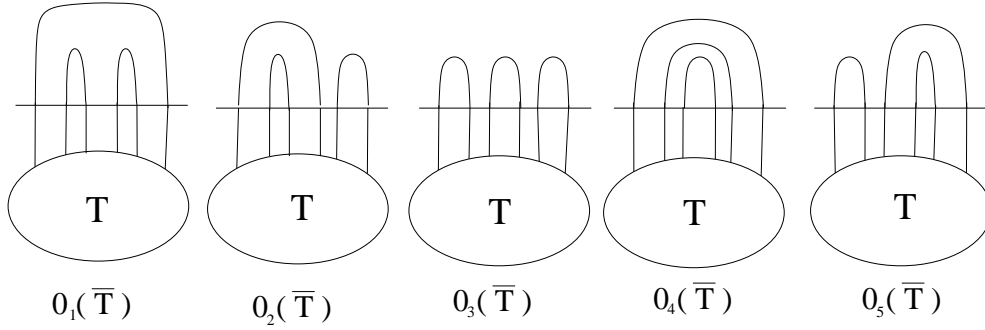


FIGURE 7.

A link diagram L is the 0_i -closure of a rational 3-tangle diagram T , denoted by $0_i(\overline{T})$, if L is obtained from T by connecting the six endpoints of T as the pattern shown above. (See Figure 7.)

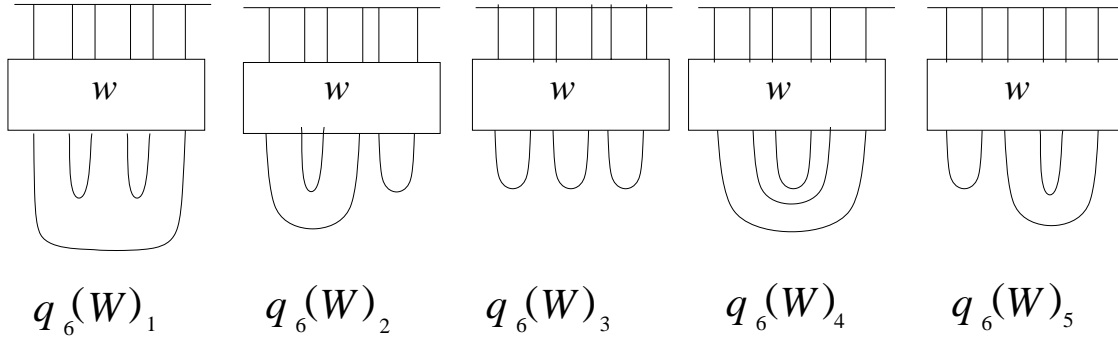


FIGURE 8.

Suppose that T has a 6-plat presentation $q_6(w)$. Then, let $w = \sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_{n-1}}^{\epsilon_{n-1}} \sigma_{k_n}^{\epsilon_n}$ for some non-zero integers ϵ_i ($1 \leq i \leq n$), where $k_i \in \{1, 2, 3, 4, 5\}$.

Now, we define $q_6(w)_i$ which is a 6-plat presentation of a rational 3-tangle T under the connectivity pattern of bottom endpoints induced by 0_i as in Figure 8. For example, $q_6(w)_3 = q_6(w)$. Then we say that a rational 3-tangle \mathbb{T} is a *6-plat tangle* if \mathbb{T} is isotopic to a tangle \mathbb{T}' and the projection of \mathbb{T}' onto the xy -plane is a 6-plat presentation $q_6(w)_i$ for some i . Then, we say that a 6-plat tangle diagram is reduced alternating if the 6-plat presentation is reduced alternating.

We note that each rational 3-tangle is a 6-plat tangle. (Refer to [9] and [10].) However, the set of all reduced alternating 6-plat tangle diagrams is a proper subset of the collection of reduced alternating rational 3-tangle diagrams.

In order to cover all reduced alternating rational 3-tangle diagrams, we define σ_6 as in Figure 9. Generally, the defined generators σ_i ($1 \leq i \leq 5$) also can be defined from the twisting obtained by the extension to B^3 of the half Dehn twists between adjacent endpoints on ∂B^3 . (Refer to [9].)

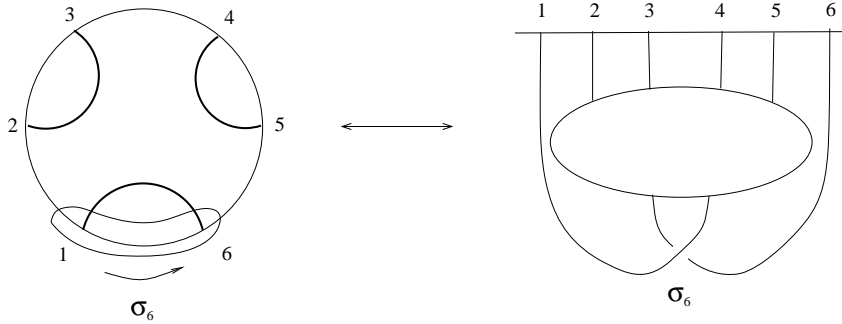


FIGURE 9.

Let $|T|$ be the minimal crossing number of the diagram T . We say that a crossing is *closest to S^2* if at least two arcs of the four end arcs from the crossing directly meet the horizontal line which stands for S^2 without any crossing. Then we have the following lemma.

Lemma 4.2. *Suppose that T is a reduced alternating rational 3-tangle diagram. If $|T| \geq 2$ then $0_i(\overline{T})$ is a reduced alternating link diagram for some i .*

Proof. First of all, we claim that if a crossing of $0_i(\overline{T})$ is not a closest crossing then the crossing cannot be removed by the first Reidemeister move. If a crossing is not closest one and the crossing is removed by Type I Reidemeister move, then the self crossing should be removed below S^2 . However, T is reduced alternating. This shows this claim.

Suppose that T has a closest crossing to S^2 which is obtained by $\sigma_1^{\pm 1}$ as in Figure 10.

First assume that T does not have any other closest crossing except the crossing.

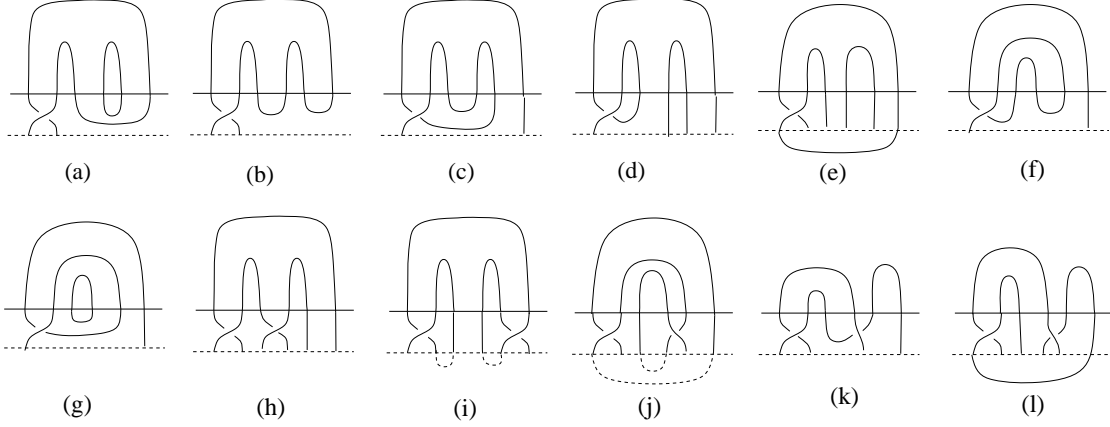


FIGURE 10.

Consider 0_1 -closure of T . Then there are five cases (a)–(e) which are not reduced alternating.

We note that the cases (a), (b) and (c) are impossible since $|T| \geq 2$ and T is a reduced alternating rational 3-tangle diagram. In the case (d) of Figure 10, take 0_3 -closure of T instead of 0_1 -closure. Then, we have the diagrams (f) and (g) to have the crossing removed by the first Reidemeister move. However, they are also impossible since $|T| \geq 2$ and T is a reduced alternating rational 3-tangle diagram. Therefore, $0_3(\overline{T})$ should be reduced alternating. Similarly, in the case (e) of Figure 10, we can check that $0_2(\overline{T})$ should be reduced alternating.

Therefore, $0_i(\overline{T})$ is reduced alternating for some i .

Now, assume that T does have another closest crossing to S^2 possibly obtained by $\sigma_3^{\pm 1}$ or $\sigma_5^{\pm 1}$.

Then consider $0_1(\overline{T})$ as in the diagram (h) and (i) of Figure 10.

In the diagram (h), take 0_2 -closure to T . Then $0_2(\overline{T})$ should be reduced alternating since T is a reduced alternating rational 3-tangle diagram. If we have a case (i), then take 0_4 -closure of T . Then, we also note that $0_4(\overline{T})$ is a reduced alternating link diagram.

It is clear that $0_2(\overline{T})$ is reduced alternating if the three crossings by $\sigma_1^{\pm 1}$, $\sigma_3^{\pm 1}$ and $\sigma_5^{\pm 1}$ are closest crossings to S^2 .

At last, we assume that T does have another closest crossing to S^2 obtained by $\sigma_4^{\pm 1}$ as the diagram (j). In order to have a crossing removed by the first Reidemeister move, at least one of the dotted arcs is a real arc. However, by taking 0_2 -closure of T as in the diagrams (k) and (l), we note that the diagrams are impossible since T is reduced alternating. So, we can eliminate this case as well.

Therefore, we just show that if T has a closest crossing to S^2 which is obtained by $\sigma_1^{\pm 1}$ then $0_i(\overline{T})$ is reduced alternating for some i .

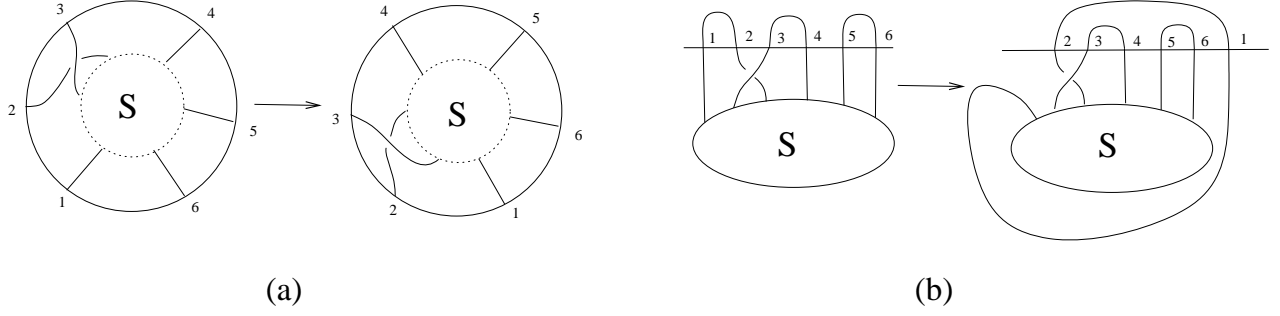


FIGURE 11.

If T has a closest crossing to S^2 which is obtained by $\sigma_2^{\pm 1}$, we modify the previous diagrams by rotating 60° counterclockwise in the disk model as the diagram (a) of Figure 11. Then this case also can be proved by the previous arguments. (Refer to the diagram (b) of Figure 11.)

Similarly, for the rest of cases to prove this lemma, we modify the previous cases by rotating a multiple of 60° in the disk model of tangle diagrams.

This completes the proof. □

Lemma 4.3. *If $|T| = 1$ Then $X_{0_i(\overline{T})} = (-a^2 - a^{-2})^{(t-1)}$, where $i \in \{1, 2, 3, 4, 5\}$ and t is the number of components of $0_i(\overline{T})$.*

Proof. Consider the diagram in Figure 12.

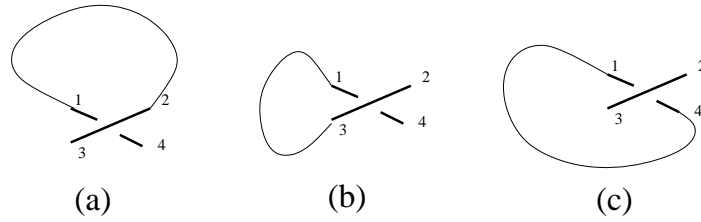


FIGURE 12.

Since T has only one crossing, the cases (a) and (b) in Figure 12 are possible. If we have a diagram (c) of Figure 12, it is impossible to connect 2 and 3. Also, the only crossing of T should be removed by the first Reidemeister move. This completes the proof of Lemma 4.3. □

Lemma 4.4. *Suppose that T and T' are reduced alternating rational 3-tangle diagrams. If $|T| = 1$ and $|T'| \geq 2$, then $v_T \neq v_{T'}$.*

Proof. First, we note that $0_i(\overline{T})$ has at most two trivial links (two split components) since $|T| = 1$ for all i . (Refer Figure 12.) By Theorem 3.1 and the third axiom for the Kauffman bracket, we note that $M(X_{0_i(\overline{T})}) - m(X_{0_i(\overline{T})}) \leq 4$.

We also note that, by Lemma 4.2, there exists 0_j -closure of T' so that $0_j(\overline{T'})$ is a reduced alternating link diagram. Therefore, the minimal crossing number of $0_j(\overline{T'})$ for some j is greater than or equal to 2. This implies that $M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})}) \geq 8$. However, if $v_T = v_{T'}$, then $4 \geq M(X_{0_j(\overline{T})}) - m(X_{0_j(\overline{T})}) = M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})}) \geq 8$ since the writhe of \overline{T} and $\overline{T'}$ cannot change $M(X_{0_j(\overline{T})}) - m(X_{0_j(\overline{T})})$ and $M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})})$. It makes a contradiction.

Therefore, $v_T \neq v_{T'}$. □

We remark that Eliahou and Kauffman [4] found an infinite number of 2-component links with Kauffman polynomial equal to $-a^2 - a^{-2}$. However, every element of the families is not (reduced) alternating by Theorem 3.1.

Lemma 4.5. *Suppose that \mathbb{T} and \mathbb{T}' are rational 3-tangles and T and T' are rational 3-tangle diagrams of \mathbb{T} and \mathbb{T}' respectively. If T and T' are reduced alternating rational 3-tangle diagrams and $|T| \leq |T'| \leq 1$, then $\mathbb{T} \approx \mathbb{T}'$ if and only if $v_T = v_{T'}$.*

Proof. We note that $v_T, v_{T'} \in \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), \dots, (0, 0, 0, 0, 1)\}$ if $|T| = |T'| = 0$. They are distinguished by the trivial rational 3-tangle type.

We also note that $v_T, v_{T'} \in \{(a^{\pm 1}, a^{\mp 1}, 0, 0, 0), (a^{\pm 1}, 0, 0, a^{\mp 1}, 0), (a^{\pm 1}, 0, 0, 0, a^{\mp 1}), (0, a^{\pm 1}, a^{\mp 1}, 0, 0), (0, 0, a^{\pm 1}, a^{\mp 1}, 0), (0, 0, a^{\pm 1}, 0, a^{\mp 1})\}$ if $|T| = |T'| = 1$. They are also distinguished by the rational 3-tangle type with $|T| = |T'| = 1$.

This completes the proof. □

Lemma 4.6. *Suppose that T and T' are reduced alternating rational 3-tangle diagrams with $v_T = (-a^{-3})^k v_{T'}$ for some k and $|T| \geq |T'| \geq 2$. Then both T and T' are positive (negative) reduced alternating, where positive (negative) alternating diagram means that the crossings of the diagram are obtained only by $\sigma_1, \sigma_3, \sigma_5, \sigma_2^{-1}, \sigma_4^{-1}$ or σ_6^{-1} ($\sigma_1^{-1}, \sigma_3^{-1}, \sigma_5^{-1}, \sigma_2, \sigma_4$ or σ_6).*

Proof. For a contradiction, assume that T is positive reduced alternating and T' is negative reduced alternating. Then, consider the diagrams of Figure 13.

We have T'_1 by having crossings in the cylinder S so that T'_1 is isotopic to a trivial tangle diagram as in the second diagram of Figure 13. We note that the crossings in S are obtained only by $\sigma_1, \sigma_3, \sigma_5, \sigma_2^{-1}, \sigma_4^{-1}$ or σ_6^{-1} .

Then attach the same cylinder S to T to have T_1 as in Figure 13. We note that $v_{T_1} = (-a^{-3})^k v_{T'_1}$ for some k , where $v_{T'_1} \in \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), \dots, (0, 0, 0, 0, 1)\}$.

Clearly, the minimal crossing number of T'_1 is zero. If T is a connected tangle diagram, then $|T_1| = |T| + |T'| \geq 4$ since T_1 is a reduced alternating rational 3-tangle diagram. So, $|T_1| \geq 4$.

We note that $0_i(\overline{T'_1})$ has at most three trivial components. So, $M(X_{0_i(\overline{T'_1})}) - m(X_{0_i(\overline{T'_1})}) \leq 8$ for all i . By Lemma 4.2, there exists a 0_k -closure of T_1 so that $0_k(\overline{T_1})$ is a reduced alternating

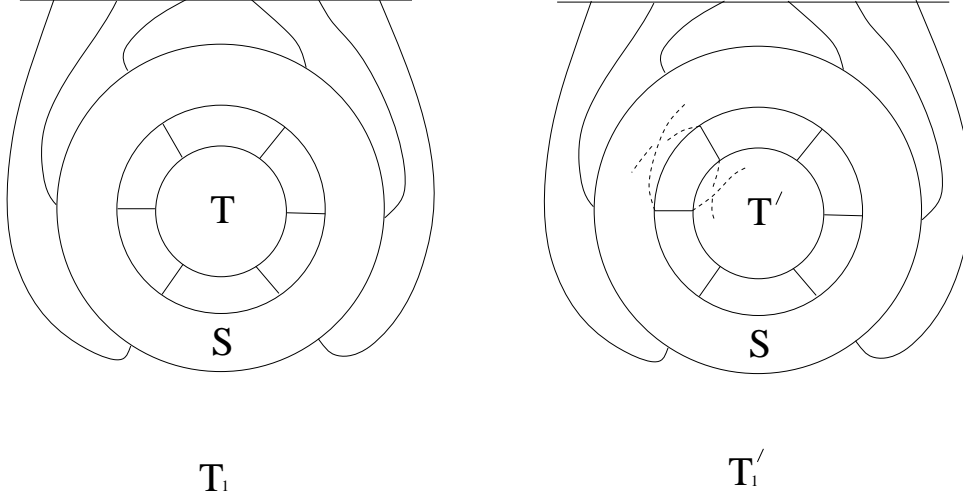


FIGURE 13.

link diagram with $|0_k(\overline{T_1})| \geq 4$. So, $M(X_{0_k(\overline{T_1})}) - m(X_{0_k(\overline{T_1})}) \geq 16$ for some k .

If $v_{T_1} = (-a^{-3})^l v_{T'_1}$ for some integer l , then we note that $X_{0_k(\overline{T_1})} = (-a^{-3})^s X_{0_k(\overline{T'_1})}$ for some s by considering the writhes of T_1 and T'_1 . This implies that $16 \leq M(X_{0_k(\overline{T_1})}) - m(X_{0_k(\overline{T_1})}) = M(X_{0_k(\overline{T'_1})}) - m(X_{0_k(\overline{T'_1})}) \leq 8$. This makes a contradiction. Therefore, $v_{T_1} \neq (-a^{-3})^l v_{T'_1}$ for any integer l .

If T is not a connected tangle diagram, then the only one possible case to have the condition $v_{T_1} = (-a^{-3})^l v_{T'_1}$ for some integer l is when $2 \leq |T'| \leq |T| \leq 3$ since if $|T| > 3$ then $|T_1| \geq 4$. So, we note that $2 \leq |T_1| \leq 3$. Then there exist i so that $0_i(\overline{T_1})$ is a reduced alternating by Lemma 4.2. However, there is no such link L so that $|L| = 2$ or 3 and $X_L = (-a^2 - a^{-2})^m$ for some integer m . (Refer to KnotInfo by Livingston/Cha.) Therefore, $v_{T_1} \neq (-a^{-3})^l v_{T'_1}$ for any integer l .

Both cases contradict the assumption that $v_T = (-a^{-3})^l v_{T'}$ for some l and this completes the proof. \square

Lemma 4.7. *Suppose that T and T' are reduced alternating rational 3-tangle diagrams. If $v_T = (-a^{-3})^k v_{T'}$ for some k , then $|T| = |T'|$.*

Proof. Suppose that $|T| \geq |T'| \geq 2$. The other cases are clear by Lemma 4.4 and Lemma 4.5. Assume that $|T| > |T'|$ for a contradiction. We note $X_{0_i(\overline{T})} = (-a^{-3})^k X_{0_i(\overline{T'})}$ for all i and some k . Otherwise $v_T \neq (-a^{-3})^k v_{T'}$. We also know that there exists j so that $0_j(\overline{T})$ is a reduced alternating link diagram by Lemma 4.2.

In order to have $X_{0_j(\overline{T})} = (-a^{-3})^k X_{0_j(\overline{T'})}$, we should have three conditions (1) $|T| = |T'| + 1$, (2) T is a connected tangle diagram and (3) T' has one separated arc. We note that T' cannot have three separated trivial arcs since $|T'| \geq 2$. Also, T' cannot be a connected tangle diagram. If T' is a connected tangle diagram then we also have $M(X_{0_j(\overline{T})}) - m(X_{0_j(\overline{T})}) > M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})})$ for some j since $|T| > |T'|$. (Refer to Theorem 2 of [12].) This

implies that $X_{0_j(\bar{T})} \neq (-a^{-3})^k X_{0_j(\bar{T}')} for some j . So, T' has one separated arc. Then we note that $|T| = |T'| + 1$. If $|T| > |T'| + 1$ then we also have $M(X_{0_j(\bar{T})}) - m(X_{0_j(\bar{T})}) > M(X_{0_j(\bar{T}')} - m(X_{0_j(\bar{T}'))}$ for some j and it makes a contradiction. At last, if T is not a connected tangle diagram then there exist j so that $0_j(\bar{T})$ is a reduced alternating link having two separated links. Then we conclude that $M(X_{0_j(\bar{T})}) - m(X_{0_j(\bar{T})}) > M(X_{0_j(\bar{T}')} - m(X_{0_j(\bar{T}'))}$ for some j since $|T| > |T'|$. Then we have $X_{0_j(\bar{T})} \neq (-a^{-3})^k X_{0_j(\bar{T}')$.$

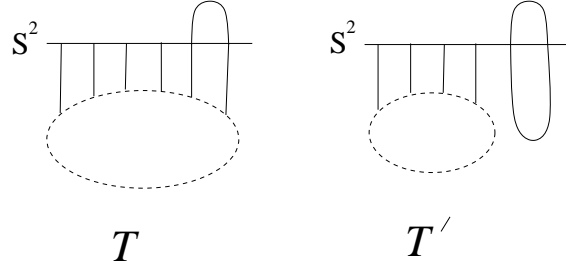


FIGURE 14.

Now, we take T and T' so that $|T| + |T'|$ is minimal. Now, consider the diagrams of Figure 14. The rest of cases are obtained from a multiple of 60° rotation in the disk model which changes the positions of the endpoints.

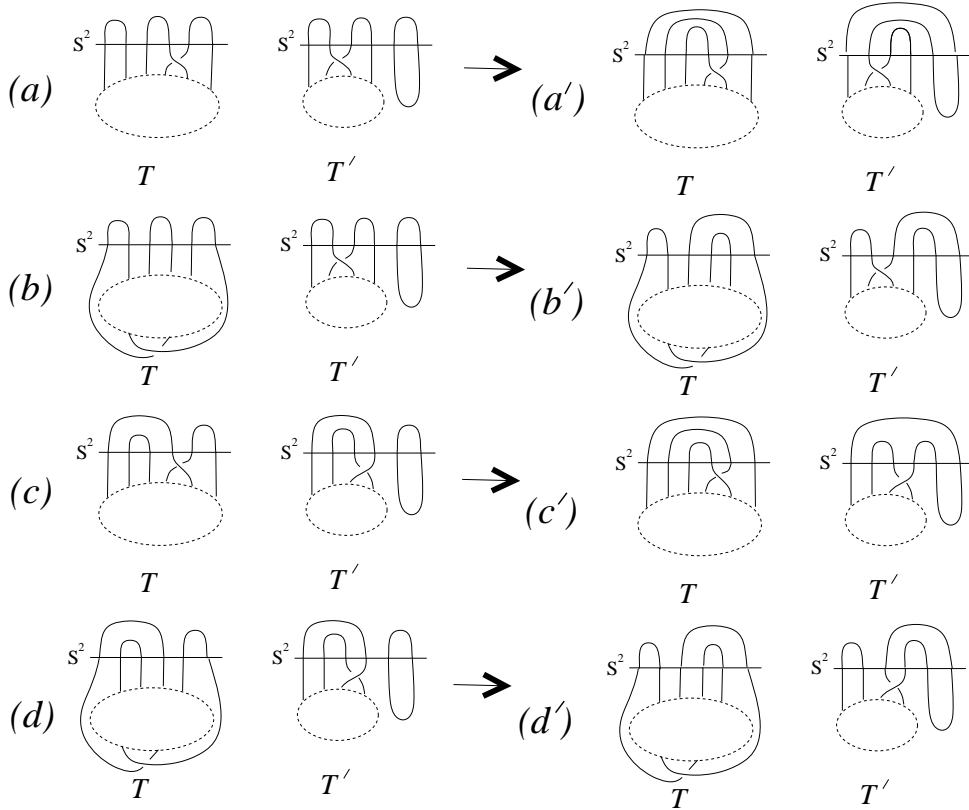


FIGURE 15.

For the 0_j -closure of T to have a connected reduced alternating link, one of the connectivity is fixed as in the diagrams of Figure 14. Otherwise, $0_j(\overline{T'})$ is also a connected link. This implies that $M(X_{0_j(\overline{T})}) - m(X_{0_j(\overline{T})}) > M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})})$ for some j since $|T| = |T'| + 1$. So, it makes a contradiction.

Then we have only four possible cases (a) – (d) for this as in Figure 15. Otherwise, either $0_j(\overline{T})$ is not reduced alternating or $|T| + |T'|$ is not minimal. We also note that both $0_j(\overline{T})$ and $0_j(\overline{T'})$ are reduced alternating. The cases (a) and (c) assume that the given crossing in T is a closest crossing to S^2 . and the cases (b) and (d) assume that the given crossing in T is the only one closest crossing to S^2 .

With a similar argument in Lemma 4.2, we can show that the closure of T in diagrams (a') – (d') is also a reduced alternating link. Also, we see that the same closure of T' in diagrams (a') – (d') is now a connected link. By using the condition that $|T| = |T'| + 1$, we have $M(X_{0_j(\overline{T})}) - m(X_{0_j(\overline{T})}) > M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})})$ for some j . Therefore, $X_{0_j(\overline{T})} \neq (-a^{-3})^k X_{0_j(\overline{T'})}$. This makes a contradiction.

So, it is impossible to have T and T' to satisfy the given three conditions.

Therefore, we conclude that $|T| = |T'|$. □

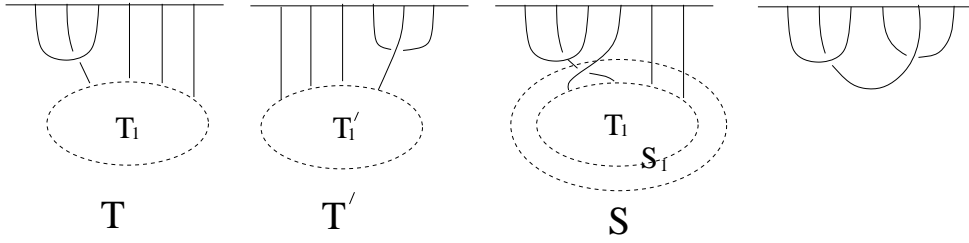


FIGURE 16.

Lemma 4.8. *Suppose that \mathbb{T} and \mathbb{T}' are rational 3-tangles with reduced alternating rational 3-tangle diagrams T and T' as in Figure 16, where T_1 and T'_1 are rational 2-tangle diagrams. If $v_T = (-a^{-3})^k v_{T'}$ for some k , then $\mathbb{T} \approx \mathbb{T}'$.*

Proof. Let $\langle T_1 \rangle = f(a) \langle T_0 \rangle + g(a) \langle T_\infty \rangle$ and $\langle T'_1 \rangle = f'(a) \langle T_0 \rangle + g'(a) \langle T_\infty \rangle$. Then for A, B, C and D of Figure 17, we have the relations $\langle T \rangle = f(a)A + g(a)B$ and $\langle T' \rangle = f'(a)C + g'(a)D$.

We also note that $A = a \langle 0_5 \rangle + a^{-1} \langle 0_1 \rangle$, $B = a \langle 0_3 \rangle + a^{-1} \langle 0_2 \rangle$, $C = a^{-1} \langle 0_1 \rangle + a \langle 0_2 \rangle$ and $D = a^{-1} \langle 0_5 \rangle + a \langle 0_3 \rangle$.

Therefore, $\langle T \rangle = f(a)A + g(a)B = f(a)(a \langle 0_5 \rangle + a^{-1} \langle 0_1 \rangle) + g(a)(a \langle 0_3 \rangle + a^{-1} \langle 0_2 \rangle) = a^{-1}f(a) \langle 0_1 \rangle + a^{-1}g(a) \langle 0_2 \rangle + ag(a) \langle 0_3 \rangle + af(a) \langle 0_5 \rangle$ and $\langle T' \rangle = f'(a)C + g'(a)D = f'(a)(a^{-1} \langle 0_1 \rangle + a \langle 0_2 \rangle) + g'(a)(a^{-1} \langle 0_5 \rangle + a \langle 0_3 \rangle) =$

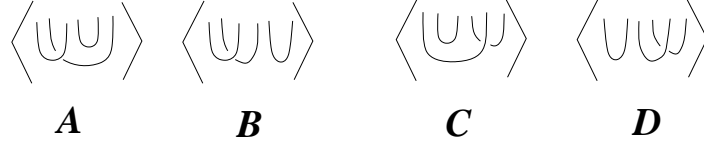


FIGURE 17.

$$a^{-1}f'(a) < 0_1 > +af'(a) < 0_2 > +ag'(a) < 0_3 > +a^{-1}g'(a) < 0_5 >.$$

This implies that $g(a) = a^2f(a)$ and $(f(a), g(a)) = (f'(a), g'(a))$. Actually, we will conclude that $T_1 = T'_1$ which has only one crossing.

Now consider the rational 3-tangle diagram S which contains the rational 2-tangle diagram S_1 as the third diagram of Figure 16, where the inner circle is for T_1 and the outer circle is for S_1 .

We have $\langle S_1 \rangle = f(a)(a \langle T_0 \rangle + a^{-1} \langle T_\infty \rangle + g(a)(-a^{-3}) \langle T_\infty \rangle) = af(a) \langle T_0 \rangle + (a^{-1}f(a) - a^{-3}g(a)) \langle T_\infty \rangle = af(a) \langle T_0 \rangle$ since $f(a) = a^2g(a)$.

We note that the Kauffman bracket vector of S_1 is $(af(a), 0)$.

In Theorem 2.2, if we consider the determinant of the matrix A_{S_1} then we have $af(a)h(a) = (a^2)^l$ for some integer l and a Laurent polynomial $h(a)$, where A_{S_1} is the matrices product for calculating the bracket vector of S_1 . This implies that $f(a) = a^n$ for some integer n . So, the Kauffman bracket vector of S_1 is $(a^{n+1}, 0)$. By a similar argument as in Theorem 3.2, we conclude that $|S_1| = 0$. So, both cases have the same diagram as the last diagram of Figure 16.

This completes the proof. □

Now, we want to prove another direction of the main theorem.

Theorem 4.9. *Suppose that \mathbb{T} and \mathbb{T}' are rational 3-tangles with reduced alternating rational 3-tangle diagrams T and T' respectively. If $v_T = (-a^{-3})^k v_{T'}$ for some k , then $\mathbb{T} \approx \mathbb{T}'$.*

Proof. Suppose that there exists reduced alternating rational 3-tangle diagrams T and T' so that $v_T = (-a^{-3})^k v_{T'}$ for some k but $\mathbb{T} \not\approx \mathbb{T}'$.

Then we choose a pair of T and T' satisfying the previous condition and $|T| = |T'|$ is minimal.

We note that both T and T' are positive (negative) reduced alternating by Lemma 4.6.

Also, we note that the closest crossings of T and T' to S^2 are obtained by different σ_i . Otherwise, by removing the common closest crossings of T and T' , we can get T_1 and T'_1 . Then we note that $v_{T_1} = (-a^{-3})^k v_{T'_1}$. So, it contradicts the assumption that $|T| = |T'|$ is minimal.

By Lemma 4.4 and Lemma 4.5, if there exists a such example then $|T| = |T'| \geq 2$.

If any of T and T' is not a connected tangle, possibly T , then take 0_j -closure of T to have a reduced alternating link for some j which has a separated trivial knot. Then we note that $0_j(\overline{T'})$ is also a reduced alternating link which has a separated trivial knot. Otherwise, we have $M(X_{0_j(\overline{T})}) - m(X_{0_j(\overline{T})}) > M(X_{0_j(\overline{T'})}) - m(X_{0_j(\overline{T'})})$. It contradicts the condition that $v_T = (-a^{-3})^k v_{T'}$ for some k since $X_{0_j(\overline{T})} \neq (-a^{-3})^k X_{0_j(\overline{T'})}$. So, it is enough to consider rational 2-tangles by ignoring the common separated trivial arc of T and T' . Then by using Corollary 3.3 we can show that if $v_T = (-a^{-3})^k v_{T'}$ for some k , then $\mathbb{T} \approx \mathbb{T}'$.

Now, we start to consider the possible cases of T .

Case 1: Assume that a crossing of T by $\sigma_1^{\pm 1}$ is the only one closest crossing to S^2 .

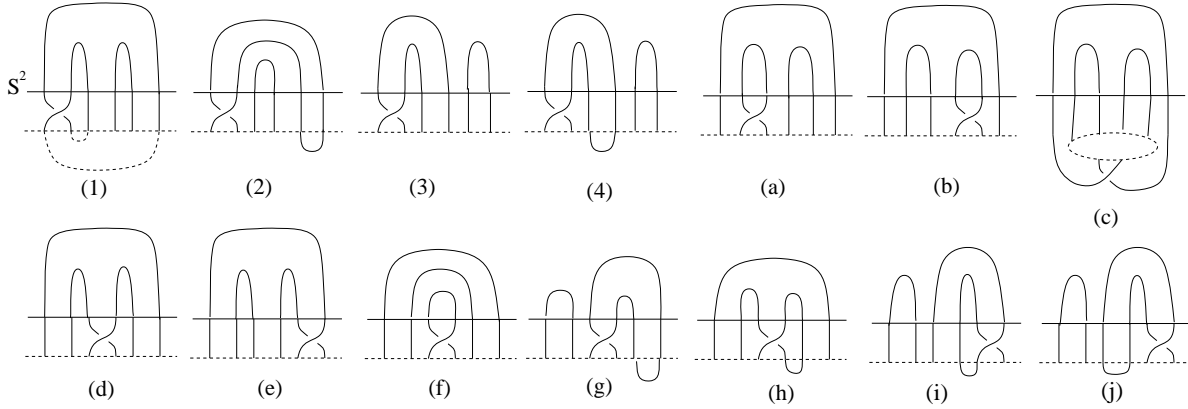


FIGURE 18.

Then 0_1 -closure of T makes a reduced alternating link diagram if the diagram (1) of Figure 18 is not the diagram of T since $|T| \geq 2$ and T is a reduced alternating tangle diagram.

First, assume that the diagram (1) is not the diagram of T .

We note that 0_1 -closure of T' also needs to be reduced alternating link diagram if the diagram (1) of Figure 18 is not the diagram of T . Otherwise, $X_{0_1(\overline{T})} \neq (-a^{-3})^k X_{0_1(\overline{T'})}$ for any k and it makes a contradiction.

We note that the diagrams (a), (b) and (c) are not reduced alternating. So, the closest crossing to S^2 of T' is by either $\sigma_3^{\pm 1}$ or $\sigma_5^{\pm 1}$. (See (d) and (e).)

Suppose that the closest crossing to S^2 of T' is by $\sigma_3^{\pm 1}$ as the diagram (d) of Figure 18.

Then we consider 0_4 -closure of T as the diagram (2) of Figure 18. If the diagram is a reduced alternating link diagram, then we have a contradiction because $0_4(\overline{T'})$ is not reduced alternating (See (f)). So, T does not have any crossing with the rightmost string of T as the

diagram (2). Now, consider 0_5 -closure of T . Then it is clear that $0_5(\overline{T})$ is not reduced alternating link diagram. So, we note that $0_5(\overline{T'})$ is also not reduced alternating link diagram. Otherwise, $X_{0_i(\overline{T})} \neq (-a^{-3})^k X_{0_i(\overline{T'})}$ for any k since $|T| = |T'|$ and it makes a contradiction. So, we have a diagram (g) or (h) for T' . Otherwise, $0_5(\overline{T'})$ is a reduced alternating link diagram, but $0_5(\overline{T})$ is not. We may consider the case that $\sigma_5^{\pm 1}$ also can be a closest crossing to S^2 since we did not cover the case. However, in this case, we note that $0_5(\overline{T'})$ is reduced alternating. This makes a contradiction too. If (g) is the diagram of T' then both T and T' have the rightmost string with no crossing. Also, the other two strings of each make a 4-plat presentation since they make a rational 2-tangle diagram. By Corollary 3.3, we note that $T = T'$ if $v_T = (-a^{-3})^k v_{T'}$ for some k . If (h) is the diagram of T' then consider 0_1 -closure of T' . Then we see that $0_1(\overline{T'})$ is not reduced alternating link diagram, but $0_1(\overline{T})$ is reduced alternating. This makes a contradiction.

Suppose that the closest crossing to S^2 of T' is by $\sigma_5^{\pm 1}$ as the diagram (e) of Figure 18.

Then, we note that $0_2(\overline{T})$ is not reduced alternating since $0_2(\overline{T'})$ is not reduced alternating. So, the second string of T cannot have any crossing as the diagram (4) of Figure 18. We note that $0_5(\overline{T'})$ is not reduced alternating since $0_5(\overline{T})$ is not. So, we have $0_5(\overline{T'})$ as the diagrams (i) and (j) of Figure 18. If we have the diagram (i) then we note that $0_1(\overline{T'})$ is not reduced alternating. This makes a contradiction since $0_1(\overline{T})$ is reduced alternating. If we have the diagram (j) then the second string of T' also does not have any crossing. By Corollary 3.3, we know that $T = T'$ if $v_T = (-a^{-3})^k v_{T'}$ for some k .

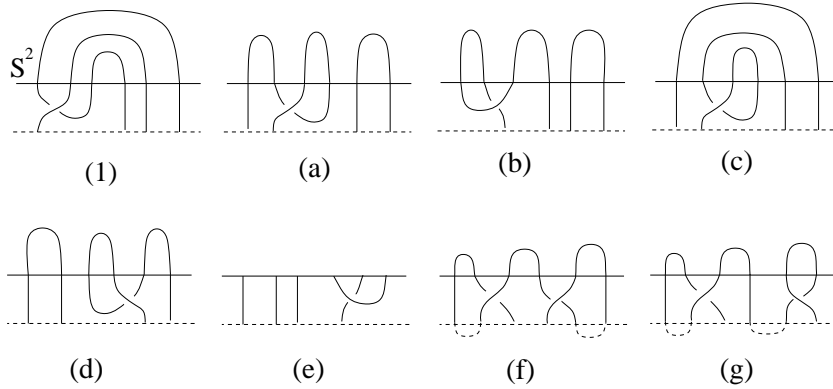


FIGURE 19.

Now, assume that the diagram (1) of Figure 18 is the diagram of T . It is enough to consider the case that the smaller dotted arc is a real arc in the diagram (1) since the other case is a symmetry case of this.

Then we note that 0_4 -closure of T is now reduced alternating as the diagram (1) in Figure 19 since $|T| \geq 2$ and T is reduced alternating. This implies that the closest crossing to S^2 of T' is either by $\sigma_2^{\pm 1}, \sigma_4^{\pm 1}, \sigma_5^{\pm 1}$ or $\sigma_6^{\pm 1}$ since $0_4(\overline{T'})$ also should be reduced alternating.

First, assume that a crossing of T' by $\sigma_2^{\pm 1}$ is the only one closest crossing to S^2 .

Since $0_3(\overline{T})$ is not reduced alternating, we note that $0_3(\overline{T'})$ is also not reduced alternating. So, the diagram of T' is either (a) of (b) of Figure 19.

If (a) is the diagram of T' then $0_4(\overline{T'})$ should be reduced alternating since $0_4(\overline{T})$ does. However, it is not reduced alternating.

(b) is also not the diagram of T' since the crossing by $\sigma_2^{\pm 1}$ is also a crossing by $\sigma_1^{\mp 1}$ in this case.

Now, assume that a crossing of T' by $\sigma_4^{\pm 1}$ is the only one closest crossing to S^2 .

Since $0_3(\overline{T})$ is not reduced alternating, $0_3(\overline{T'})$ should be not reduced alternating. Therefore, the diagram of T' is either (d) or (e) of Figure 19.

If (d) is the diagram of T' then $0_4(\overline{T'})$ should be reduced alternating since $0_4(\overline{T})$ does. However, it is not reduced alternating.

By Lemma 4.8, (e) is also not the diagram of T' .

The cases that a crossing of T' by $\sigma_5^{\pm 1}$ or $\sigma_6^{\pm 1}$ is the only one closest crossing to S^2 are analogous to the previous two cases.

Now, assume that crossings of T' by $\sigma_2^{\pm 1}$ and $\sigma_4^{\pm 1}$ are the closest crossings to S^2 .

Then, take 0_3 -closure of T' as the diagram (f) of Figure 19. Since $0_3(\overline{T})$ is not reduced alternating, $0_3(\overline{T'})$ is also not reduced alternating. This implies that at least one of the dotted arcs in the diagram (f) of Figure 19 is a real arc. (If a crossing by $\sigma_6^{\pm 1}$ is also a closest crossing to S^2 then clearly, $0_3(\overline{T'})$ is reduced alternating.) Both cases can be eliminated by a similar argument as in the previous cases.

Assume that crossings of T' by $\sigma_2^{\pm 1}$ and $\sigma_5^{\pm 1}$ are the closest crossings to S^2 .

Then, take 0_2 -closure of T' as the diagram (g) of Figure 19. Since $0_5(\overline{T})$ is not reduced alternating, $0_5(\overline{T'})$ is also not reduced alternating. This implies that at least one of the dotted arcs in the diagram (g) of Figure 19 is a real arc. Both cases can be eliminated by the same argument as in the previous case.

The cases having closest crossings by $\sigma_2^{\pm 1}$ and $\sigma_6^{\pm 1}$ or $\sigma_6^{\pm 1}$ and $\sigma_6^{\pm 1}$ are analogous to one of the previous cases.

Therefore, there is no counterexample when the diagram of T is the diagram (1) of Figure 18.

Case 2: Now, assume that two crossings which are obtained by $\sigma_1^{\pm 1}$ and $\sigma_3^{\pm 1}$ are closest crossings to S^2 and no other crossing can be a closest crossing to S^2 .

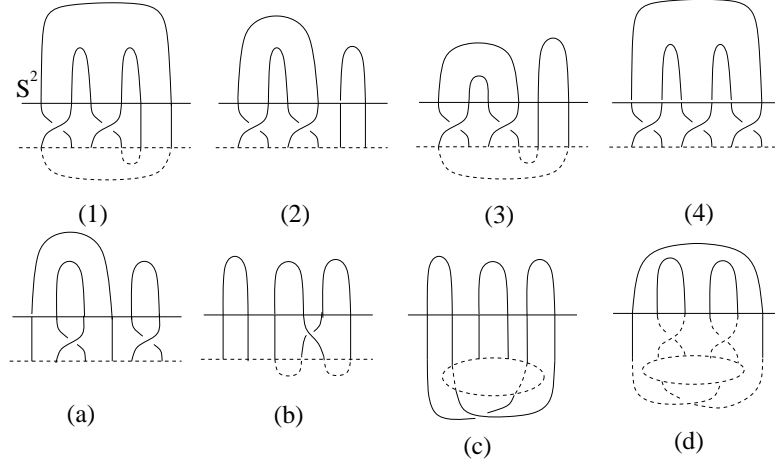


FIGURE 20.

Then 0_2 -closure of T are reduced alternating link diagrams as the diagrams (2) of Figure 20 since T is reduced alternating.

We note that T' cannot have a closest crossing by $\sigma_1^{\pm 1}$ or $\sigma_3^{\pm 1}$. Otherwise, it violates the rule that $|T| = |T'|$ is minimal. So, a closest crossing of T' is by either $\sigma_2^{\pm 1}, \sigma_4^{\pm 1}, \sigma_5^{\pm 1}$ or $\sigma_6^{\pm 1}$.

If we have a closest crossing by $\sigma_2^{\pm 1}$ or $\sigma_5^{\pm 1}$, then we see that $0_2(\overline{T'})$ is not reduced alternating as the diagram (a) of Figure 20. This makes a contradiction.

Now consider the case that a crossing by $\sigma_4^{\pm 1}$ or $\sigma_6^{\pm 1}$ is only two possible closest crossing to S^2 in T' . Then we also note that $0_1(\overline{T'})$ is not reduced alternating. (Refer to (b) of Figure 20.) So, $0_1(\overline{T})$ is also not reduced alternating. This implies that the diagram (1) of Figure 20 is the diagram of T . i.e., at least one of the dotted arcs is a real arc. Then $0_2(\overline{T})$ is reduced alternating as the diagram (3) of Figure 20.

Assume that the crossing by $\sigma_4^{\pm 1}$ is the only one closest crossing to S^2 in T' . Since $0_3(\overline{T})$ is not reduced alternating, $0_3(\overline{T'})$ should be not reduced alternating. So, at least one of the dotted arcs of the diagram (b) is a real arc. However, the right dotted arc cannot be real since $0_2(\overline{T'})$ should be reduced alternating. Also, the self crossing by the left dotted arc impossible since the crossing by $\sigma_4^{\pm 1}$ is not also a crossing by $\sigma_3^{\mp 1}$ and it make a contradiction for $|T| = |T'|$ is minimal.

Now, assume that the crossing by $\sigma_6^{\pm 1}$ is the only one closest crossing to S^2 in T' . Then we can eliminate this case by a similar argument as in the previous case. (Refer to the diagram (c) of Figure 20.)

Suppose that both $\sigma_4^{\pm 1}$ and $\sigma_6^{\pm 1}$ make a closest crossing to S^2 for T' . Then we can take 0_3 -closure of T' and we conclude that $0_3(\overline{T'})$ should be reduced alternating by a similar argument as in the previous cases. However, $0_3(\overline{T})$ is not reduced alternating. Therefore, the

second case also cannot be realized.

Case 3: Assume that three crossings which are obtained by $\sigma_1^{\pm 1}$, $\sigma_3^{\pm 1}$ and $\sigma_5^{\pm 1}$ are closest crossings to S^2 as the diagram (4) in Figure 20.

We note that a closest crossing to S^2 of T' is by either $\sigma_2^{\pm 1}, \sigma_4^{\pm 1}$ or $\sigma_6^{\pm 1}$ since $|T| = |T'|$ is minimal.

We note that 0_1 -closure of T is reduced alternating link diagram. However, $0_1(\overline{T'})$ is not a reduced alternating as the diagram (d) of Figure 20. So, this case is also impossible.

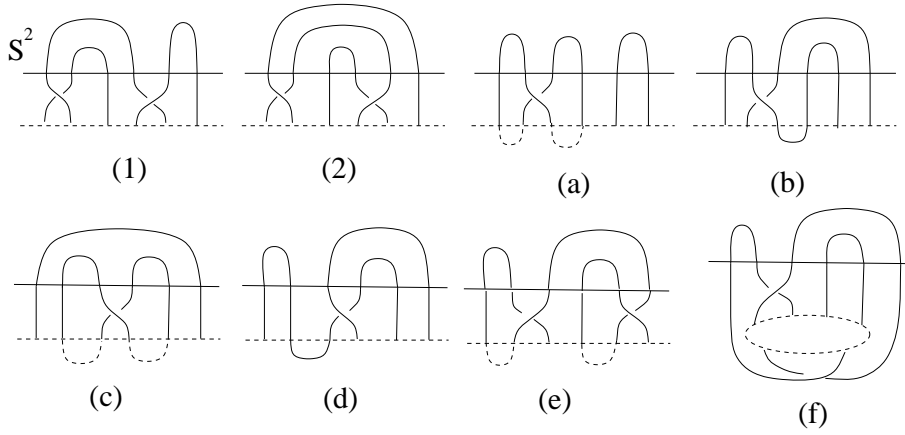


FIGURE 21.

Case 4: Assume that the two crossings which are obtained by $\sigma_1^{\pm 1}$ and $\sigma_4^{\mp 1}$ are closest crossings to S^2 as the diagrams (1) and (2) in Figure 21.

We note that at least one of the diagrams (1) and (2) is reduced alternating by Lemma 4.2. Also, we note that a closest crossing to S^2 of T' is by either $\sigma_2^{\mp 1}, \sigma_3^{\pm 1}, \sigma_5^{\pm 1}$ or $\sigma_6^{\mp 1}$.

First, assume that the crossing by $\sigma_2^{\mp 1}$ is the only one closest crossing to S^2 of T' .

Then take 0_3 -closure of T' as the diagram (a) of Figure 21. We note that one of the dotted arc should be real since $0_3(\overline{T'})$ is not reduced alternating ($0_3(\overline{T})$ is not reduced alternating), T' is reduced alternating and $|T'| \geq 2$. It is impossible to have the left dotted arc since we also can get the crossing (by $\sigma_2^{\mp 1}$) from $\sigma_1^{\pm 1}$ in this case. It contradicts the assumption that $|T| = |T'|$ is minimal. Now, assume that the right dotted arc only can be realized. Then we note that $0_5(\overline{T'})$ is reduced alternating but $0_5(\overline{T})$ is not. This makes another contradiction.

Second, assume that the crossing by $\sigma_3^{\pm 1}$ is the only one closest crossing to S^2 of T' . Consider the diagram (c) and (d). Then, we can make a contradiction by using a similar argument as in the previous case.

Now, assume that the crossings by $\sigma_2^{\pm 1}$ and $\sigma_5^{\pm 1}$ is the only two closest crossing to S^2 . We note that $0_5(\overline{T'})$ is not reduced alternating since $0_5(\overline{T})$ is not reduced alternating. So, at least one of the dotted arcs in the diagram (e) should be real arc. However, any of the arcs cannot be realized since the crossings by $\sigma_2^{\pm 1}$ or $\sigma_5^{\pm 1}$ also can be obtained from either $\sigma_1^{\pm 1}$ or $\sigma_4^{\mp 1}$ in these two cases.

Assume that the crossings by $\sigma_2^{\pm 1}$ and $\sigma_6^{\pm 1}$ is the only two closest crossing to S^2 of T' . Then $0_5(\overline{T'})$ is reduced alternating as the diagram (f) of Figure 21. However, $0_5(\overline{T'})$ is not reduced alternating. This makes a contradiction.

We note that the rest of cases also can be eliminated by a similar argument in the previous cases since we can modified each of the remaining cases into a case I already covered.

Therefore, it is impossible to have a closest crossing by $\sigma_1^{\pm 1}$ to satisfy the assumption that $v_T = (-a^{-3})^k v_{T'}$ for some k but $T \not\approx T'$.

The cases having a closest crossing to S^2 of T by $\sigma_j^{\pm 1}$ for $2 \leq j \leq 6$ are also eliminated by a similar argument as in the previous cases by considering $n \times 60^\circ$ counterclockwise rotations ($1 \leq n \leq 5$). (Refer to Figure 11.)

This completes the proof. □

Conjecture 4.10. *Suppose that T and T' are two rational 3-tangles. Then $T \approx T'$ if and only if $v_T = (-a^{-3})^k v_{T'}$ for some integer k .*

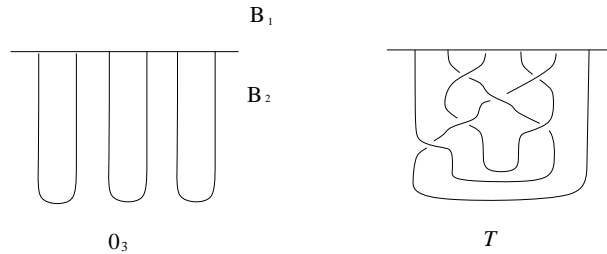


FIGURE 22.

Example : Figure 22 gives us an example of two rational alternating 3-tangles that are distinguished by the invariant. We note that $0_3(\overline{T})$ is the Borromean rings. So, if you take any two of the three strings in T then we get a trivial rational 2-tangle.

First of all, we see that $v_{0_3} = (0, 0, 1, 0, 0)$.

I found a method to calculate the Kauffman bracket of 6-plat presentations of links by using a presentation of braid group \mathbb{B}_6 into a group of 5×5 matrices. (Refer to [10].)

By the method, we have $v_T = (-a^{-6} + 3a^{-2} - 3a^2 + a^6, -2a^4 + a^8 + 1, -2a^6 + a^{10}, -a^4, -2a^4 + a^8 + 1)$.

Therefore, $v_{0_3} \neq (-a^{-3})^k v_T$ for any k .

This implies that $0_3 \not\approx T$.

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